数理解析研究所講究録
466巻 1982年 129-139

S^3上の non-singular Morse-Smale flow の閉軌道のなす link について.

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Theorem. Suppose that a non-singular Morse-Smale flow on S^3 has a single closed orbit h_0 of index 2, a single closed orbit h_{n+1} of index 0, and n closed orbits $h_1, h_2, \dots, h_n$ of index 1.

(A) If all of the closed orbits of index 1 are untwisted, then the link consisting of all closed orbits is trivial.

(B) If all of the closed orbits of index 1 are twisted, then by re-ordering $h_1, h_2, \dots, h_n$ appropriately, we find k such that

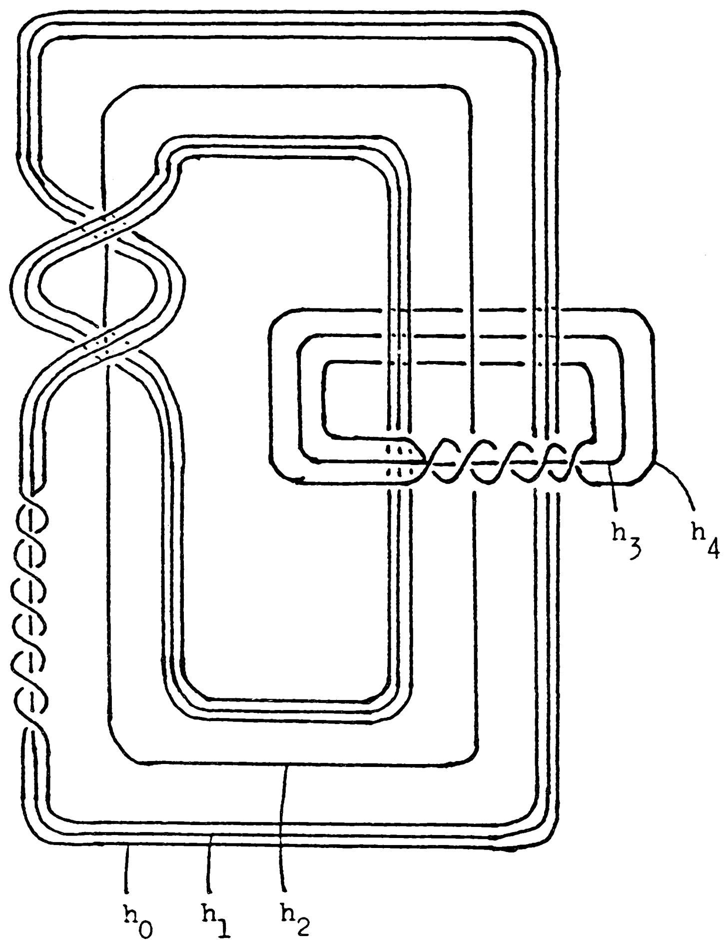
(a) h_k and h_{k+1} make the Hopf link,

(b) for any $i < k$, h_i is a $(2, p_i)$-cable of h_{i+1}, and

(c) for any $j > k$, h_{j+1} is a $(2, q_j)$-cable of h_j,

where p_i and q_j are arbitrary odd integers. (See Figure 1.)

Corollary. Suppose that a non-singular Morse-Smale flow on S^3 has three closed orbits. Then there exists just one closed orbit of index i for each i ($0 \leqq i \leqq 2$), and the closed orbits make either the trivial link or the link as in Figure 2, where h is the orbit of index 1 and L_1 is a $(2,p)$-cable of h (p = odd), and one of L_1 and L_2 is of index 0 and the other is of index 2.



(This figure shows the case n=3,k=2.)

Figure 1.

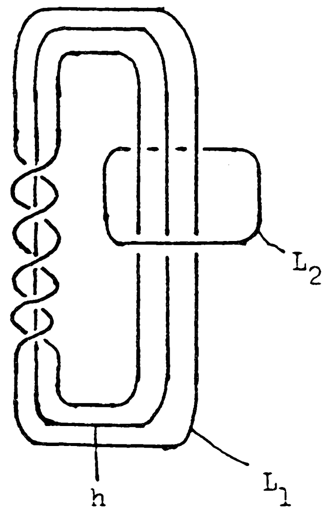


Figure 2.

The author is very grateful to Professors Itiro Tamura, Shigenori Matsumoto, Koich Yano for their advice and encouragement, Professor Nagayoshi Iwahori for his advice on Algebra, and Noriko Maruyama for her encouragement.

1 Definitions.

In this section we recall fundamental definitions and properties briefly. For further details see [F3],[M],[Sm] . On the low dimensional topology see [R].

A <u>non-singular Morse-Smale flow</u> (or an <u>NMS flow</u> for short) on a manifold M^n is a flow without fixed points which satisfies the following conditions:

(1) The non-wandering set consists entirely of finite number of closed orbits.

(2) The Poincaré map for each closed orbit is hyperbolic.

(3) If c and c' are closed orbits, then the stable manifold of c and the unstable manifold of c' intersect transversely.

Then the dimension of the unstable bundle of a closed orbit c is called the <u>index</u> of c. A closed orbit is called <u>untwisted</u> if its unstable bundle is orientable. Otherwise it is called <u>twisted</u>.

Associated to an NMS flow, we can consider a round handle decomposition of M^n.

Definition 1.1. (a) Let X^n, Y^n be manifolds. X^n is obtained from Y^n by <u>attaching a round k-handle</u> if

(1) There are disk bundles E^k_s and E^{n-k-1}_u over S^1, and

(2) an embedding $\theta : \partial E^k_s \times E^{n-k-1}_u \longrightarrow \partial Y^n$ such that $X^n \approx Y^n \bigcup_\theta (E^k_s \oplus E^{n-k-1}_u)$.

(b) The total space of $E^k_s \oplus E^{n-k-1}_u$ is called a <u>round k-handle</u>.

(c) A <u>round handle decomposition</u> for X^n is a filtration

$$X_0 \subset X_1 \subset X_2 \subset \dots \subset X_k = X ,$$

where each X_i is obtained from X_{i-1} by attaching a round handle.

D. Asimov and J. Morgan connected an NMS flow with a round handle decomposition as follows.

Proposition 1.2 [A],[M]. If a manifold M has an NMS flow, then M has a round handle decomposition whose core circles are the closed orbits of the flow. Conversely, if M has a round handle decomposition, then M has an NMS flow whose closed orbits are the core circles of round handles.

A round handle is called <u>untwisted</u> if its core circle is an untwisted closed orbit. Otherwise it is called <u>twisted</u>.

In the rest of this paper we consider an NMS flow on S^3. Then a round 1-handle H is of the form $H = E^1_s \oplus E^1_u$ and the part $\partial E^1_s \times E^1_u$ of ∂H is two copies of annuli if H is untwisted, or an annulus if H is twisted. Each annulus is mapped to a small tubular neighborhood of a circle on the boundary surface of 3-manifold. Such a circle is called the <u>attaching circle</u> of H.

2 Preliminaries.

In this section we give preliminary lemmas which are necessary to prove our main theorem. Let an NMS flow be given on S^3.

Lemma 2.1. Let U be a solid torus in S^3 such that the flow is outwardly transverse to the boundary ∂U of U.

Let H be an untwisted round 1-handle with core h, which is attached to U. Then the resulting manifold $U \cup H$ is one of the following.

(A) A solid torus with a small tubular neighborhood of an interior circle deleted.

(B) A solid torus, in which U and h are put trivially (see Figure 2.1).

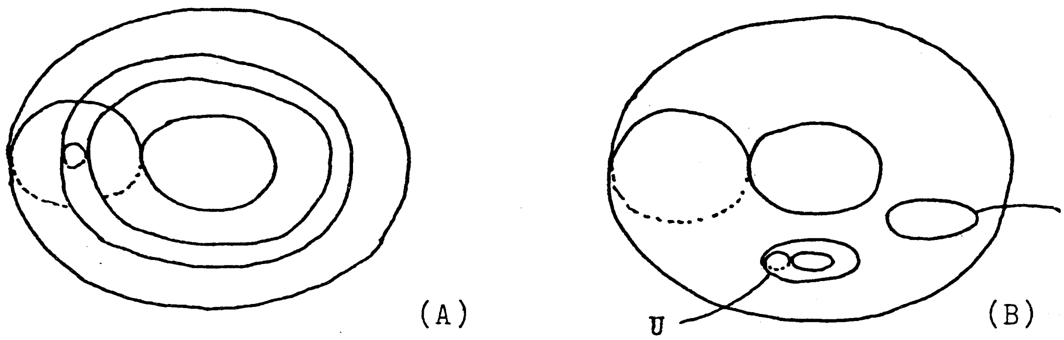


Figure 2.1.

Proof. Let K_1 and K_2 be attaching circles of H on ∂U. If $[K_1]=[K_2]=0$ in $H_1(\partial U)$ then four cases as in Figure 2.2 occur.

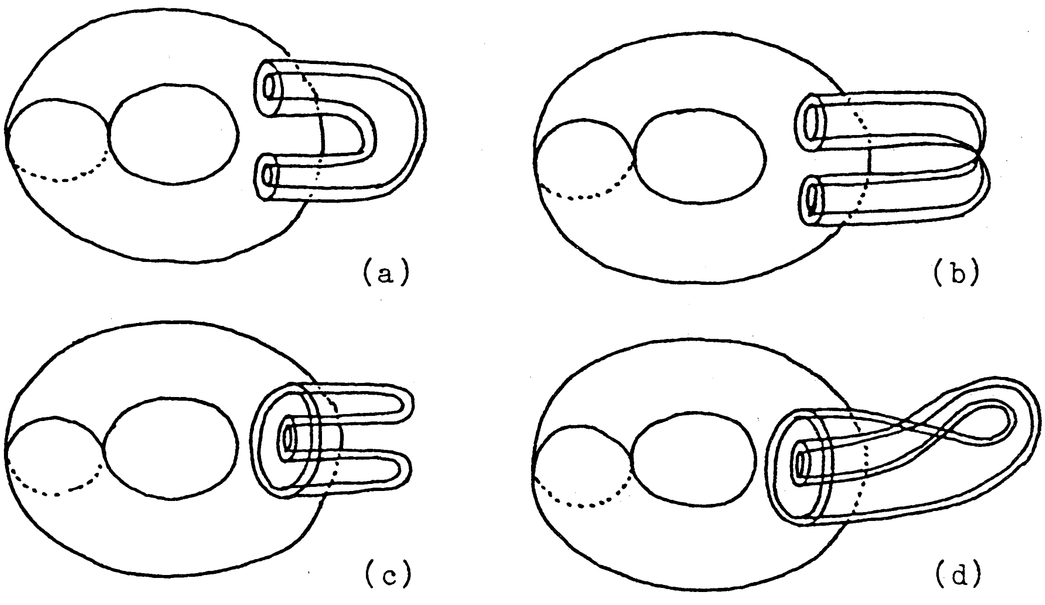


Figure 2.2.

In the case (a), the boundary of $U \cup H$ contains S^2 as an connected component. S^2 bounds a 3-ball and the flow is transverse to S^2. But such a flow in a 3-ball has a fixed point. This contradicts to our assumption. In the case (b) or (d), the Klein bottle is embedded in S^3. This is a contradiction. In the case (c), we have (A) in our Lemma. If $[K_1] = 0$ and $[K_2] = a[m_u] + b[l_u] \neq 0$, where m_u and l_u are the meridian and the longitude of U. Then $a = lk([K_2],[u]) = lk([K_1],[u]) = 0$, where u is the core of U and $lk(,)$ denotes the linking number. Thus $b = \pm 1$, and we obtain (B). If $[K_1] = [K_2] \neq 0$ it is easy to show that we have (A). This completes the proof. □

Lemma 2.2 Let U be a solid torus in S^3 such that the flow is outwardly transverse to the boundary of U. Let H be a twisted round 1-handle with core h, which is attached to U. Let $S = S^3 - (U \cup H)$. Then $U \cup H$ is one of the following.

(A) A solid torus.

(B) The exterior of a (2,p)-torus knot, and S is the tubular neighborhood of the knot, where p is an arbitrary odd integer (see Figure 2.3).

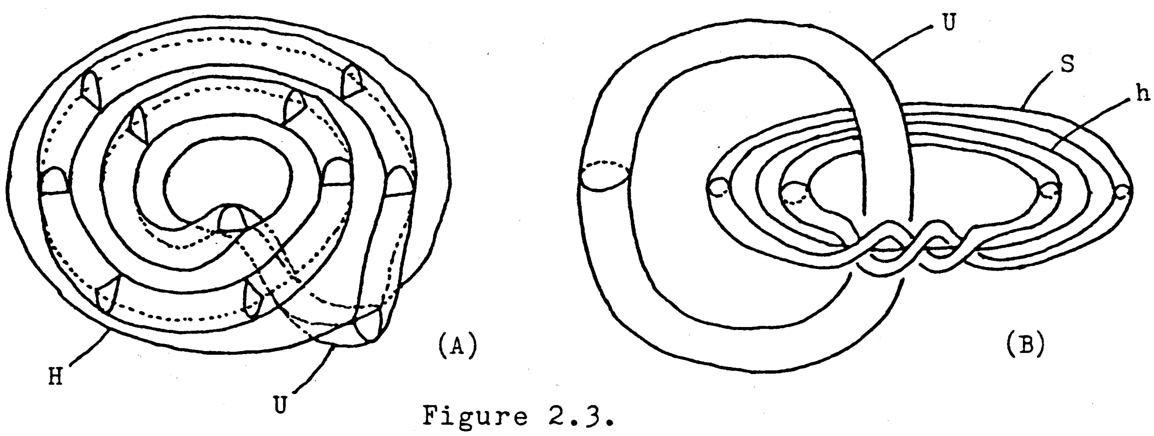


Figure 2.3.

Proof. Let K be the attaching circle of H on ∂U. Let m_u and l_u be the meridian and the longitude of U. If $[K]=0$ or $[K]=[m_u]$ in $H_1(\partial U)$, then the projective plane P^2 is embedded in S^3. This is a contradiction. Suppose $[K]=a[m_u]+b[l_u]$, where $b\neq 0$. We may assume $b>0$ and $-b/2<a<b/2$. If $b=1$, then $a=0$ and $U\cup H$ is a solid torus. This gives (A). Assume $b>1$. by the solid torus theorem ([R] p.107), the boundary torus $\partial(U\cup H)\approx T^2$ bounds a solid torus on at least one side. But

$$\pi_1(U\cup H) = \langle\ h,\ u;\ \ h^2u^b=1\ \rangle \neq \mathbb{Z}\ ,$$

where h and u are the generator of the fundamental group represented by the orbit h and the core u of U. Thus S is a solid torus and $U\cup H$ is a knot exterior. Hence $H_1(U\cup H)=\mathbb{Z}$. So b is odd. The fundamental group of the boundary $\partial(U\cup H)$ is generated by u^b and hu^x, where x is the integer such that $ax\equiv -1 \pmod{b}$ and $0<x<b$. Let $\psi\colon \partial(U\cup H)\longrightarrow \partial S$ be the inverse of the attaching map of S to $U\cup H$, and assume that the induced map $\psi_*\colon H_1(\partial(U\cup H))\longrightarrow H_1(\partial S)$ is given by

$$\psi_*([u^b]) = q[m_s]+r[l_s] \quad \text{and}$$
$$\psi_*([hu^x]) = s[m_s]+t[l_s],$$

where m_s and l_s are the meridian and the longitude of S and $qt-rs=\pm 1$. Then $\psi_*^{-1}([m_s]) = \pm(t[u^b]-r[hu^x])$. Hence we have

$$\pi_1((U\cup H)\cup S) = \langle\ h,\ u;\ \ h^2u^b=1,\ \ u^{bt}(hu^x)^{-r}=1\ \rangle\ .$$

We denote this group by G. Since $(U\cup H)\cup S=S^3$, G should be a trivial group. We prepare

Lemma 2.3. G is trivial if and only if $t=0$, $r=\pm 1$, and $x=(b\pm 1)/2$ (consequently $a=\mp 2$).

Proof of Lemma 2.3. Let
$\tilde{G} = \langle \tilde{h}, \tilde{u};\ \tilde{h}^2 = \tilde{u}^b = (\tilde{h}\tilde{u}^x)^r = 1 \rangle$.
Note that $\tilde{G}$ is a surjective image of G, and $G = \{1\}$ implies $\tilde{G} = \{1\}$. By putting $\tilde{v} = \tilde{u}^x$ and $\tilde{w} = (\tilde{h}\tilde{v})^{-1}$, $\tilde{G}$ is re-written as

$$\tilde{G} = \langle \tilde{h}, \tilde{v}, \tilde{w};\ \tilde{h}^2 = \tilde{v}^b = \tilde{w}^r = \tilde{h}\tilde{v}\tilde{w} = 1 \rangle .$$

By a famous theorem of H. Coxeter [CM], $\tilde{G} = \{1\}$ if and only if $r = \pm 1$. Thus $G = \{1\}$ implies $r = \pm 1$. Conversely, assume $r = \pm 1$. Then $G = \langle u;\ u^{2(-x \pm tb)+b} = 1 \rangle$. Hence it is necessary that $t = 0$, and $x = (b \pm 1)/2$. This completes the proof. $\square$

We continue the proof of Lemma 2.2. By Lemma 2.3, $r = \pm 1$ and $\psi_*([u^b]) = q[m_s] \pm [l_s]$. Under an appropriate diffeomorphism of S, we may assume $\psi_*([u^b]) = [l_s]$. Since the core of the part of ∂H attached to ∂S (with respect to the flow with the reversed direction) represents h^2 in $\pi_1(H)$, the attaching circle of H on ∂S is the longitude of S. Hence this case reduces to the case of $b = 1$, by reversing the direction of the flow. So $H \cup S$ is a solid torus and U is an unknotted torus. This completes the proof. $\square$

3 Proof of Theorem and Corollary.

Proof of Theorem. Let h_i be as in Theorem and H_i be a round handle with core h_i. By re-ordering h_i's if necessary, we may assume that H_{i+1} is attached to $\bigcup_{0 \leq j \leq i} H_j$. Let $H(i) = \bigcup_{0 \leq j \leq i} H_j$ ($0 \leqq i \leqq n+1$).

(A) Assume that all of $h_1, h_2, \ldots, h_n$ are untwisted.

We prove, by induction on i, that $H(i)$ is a solid torus for any $i = 0,1,\dots,n$. For $i = 0$, it is trivial. Assume that it is proved for $i < n$. Applying Lemma 2.1 to $U = H(i)$ and $H = H_{i+1}$, we have two cases of $H(i+1)$. If $H(i+1)$ is as (A) in Lemma 2.1, its boundary is the disjoint union of two tori and the flow is outwardly transverse to these tori. To make S^3 which is without boundary, we must connect these tori by $H_{i+2} \cup H_{i+3} \cup \dots \cup H_{n+1}$. Hence we can choose a simple closed curve which intersects an embedded torus (each component of the boundary of $H(i+1)$) in S^3 just one time. This leads to a contradiction. Thus $H(i+1)$ is a solid torus, and the induction is completed. Note that $H(i)$ and h_{i+1} are put trivially in $H(i+1)$. Thus it is proved inductively that $h_0, h_1, \dots, h_n$ are put trivially in the solid torus $H(n)$. Attaching H_{n+1} to $H(n)$ so that $H(n) \cup H_{n+1} = S^3$, we can prove (A) of Theorem immdediately.

(B) Assume that all of $h_1, h_2, \dots, h_n$ are twisted. Let k be the maximum number such that $H(i)$ is a solid torus for every $i = 0,1,\dots,k$. If $k = n$ then (B) of Theorem is immediate. We assume $k < n$. By reversing the direction of the flow, we may regard that H_i is attached to $H_{i+1} \cup \dots \cup H_{n+1}$. Let $S(i) = \bigcup_{i \leq j \leq n+1} H_j$. Note that $S(k+2)$ is a solid torus by Lemma 2.2. We prove that $S(i)$ is a solid torus for $i = k+1, k+2, \dots, n+1$. Let l be the minimum number so that $S(i)$ is a solid torus for every $i = l, l+1, \dots, n+1$. We will prove $l = k+1$. Since $S(k+2)$ is a solid torus, $S(k+1)$ is also s solid torus by Lemma 2.2. So $l \neq k+2$. Assume $l > k+2$. Let s be the core of the solid torus $S(l)$. Then $\pi_1(S(l)) = \langle s \rangle = \mathbb{Z}$, and

$\pi_1(S(l-1)) = \langle h_{l-1}, s;\ h_{l-1}^2 s^b = 1 \rangle$, where b is an odd integer greater than or equal to 3. Inductively we have

$$\pi_1(S(k+2)) = \langle h_{k+2}, h_{k+3}, \ldots, h_{l-1}, s;\ h_{l-1}^2 s^b = 1,\ h_{l-2}^2 = r_{l-2}, \ldots, h_{k+2}^2 = r_{k+2} \rangle,$$

where r_j is a word in $h_{j+1}, h_{j+2}, \ldots, h_{l-1}, s$. This group contains $\langle h_{l-1}, s;\ h_{l-1}^2 s^b = 1 \rangle$ as a subgroup. Hence this group is not $\mathbb{Z}$. This contradicts to the fact that $S(k+2)$ is a solid torus. Hence $l = k+1$. Now the proof of (B) of Theorem is easy.

The proof of Theorem is completed. $\square$

Proof of Corollary. J. Franks proved that the number A_i of untwisted closed orbits of index i of any NMS flow on S^3 satisfies that $A_0 \geq 1$, $A_2 \geq 1$, and $A_1 \geq \max(A_0-1, A_2-1)$. Thus there exists just one closed orbit of index i for each i. The rest of Corollary is immediately proved by Theorem. $\square$

4 Concluding remarks.

Our last aim is the complete classification on links of closed orbits of NMS flows. We give some results for our aim.

(1) The author obtained the complete classification of links of closed orbits of any NMS flow with at most 5 closed orbits on S^3. For the result, consult [Sa].

(2) K. Yano [Y] proved that any NMS flow on S^3 has at least two unknotted closed orbits.

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